Fractional Kinetic Equations Involving $p-k$-Mittag-Leffler Function

Gurmej Singh[a], Mehar Chand[b]

[a]Department of Mathematics, Mata Sahib Kaur Girls College, Talwandi sabo, Bathinda-151302, India
[b]Department of Mathematics, Baba Farid College, Bathinda-1513002, India

Abstract

We develop a new and further generalized form of the fractional kinetic equation involving generalized $p-k$-Mittag-Leffler function. The solution of the fractional kinetic equation is obtained in terms of Mittag-Leffler function. The results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

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1. Introduction and Preliminaries

The two parameter Pochhammer symbol, recently introduced by Gehlot in [1] is defined as:

Definition 1. *Let $\xi \in \mathbb{C};\ p,k \in \mathbb{R}^+ - 0;\ n \in \mathbb{N};\ \Re(\xi) > 0$, then $p-k$ Pochhammer symbol is defined as:*

$$\begin{aligned} {}_p(\xi)_{n,k} &= \left(\frac{\xi p}{k}\right)\left(\frac{\xi p}{k}+p\right)\left(\frac{\xi p}{k}+2p\right)\cdots\left(\frac{\xi p}{k}+(n-1)p\right) \\ &= \frac{{}_p\Gamma_k(\xi+nk)}{{}_p\Gamma_k(\xi)}. \end{aligned} \tag{1.1}$$

The two parameter gamma function is defined as [1]:

Definition 2. *Let $\xi \in \mathbb{C} \setminus k\mathbb{Z}^-;\ p,k \in \mathbb{R}^+ - 0;\ n \in \mathbb{N};\ \Re(\xi) > 0$, then $p-k$ Gamma function is defined as:*

$${}_p\Gamma_k(\xi) = \int_0^\infty e^{-\frac{t^k}{p}} t^{\xi-1} dt. \tag{1.2}$$

Email addresses: gurmejsandhu11@gmail.com (Gurmej Singh), mehar.jallandhra@gmail.com (Mehar Chand)

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Recently in [2], Gehlot introduced the $p-k$ Mittag-Leffler function defined as:

Definition 3. *Let* $p, k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then* $p-k$ *Mitag-Leffler function is defined as:*

$$ {}_pE^{\eta,q}_{k,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{{}_p(\eta)_{nq,k}}{{}_p\Gamma_k(n\lambda+\mu)} \frac{z^n}{n!}. \tag{1.3}$$

where ${}_p(\eta)_{nq,k}$ is two parameter Pochhammer symbol defined in equation (1.1).

1.1. Special Cases

1. On setting $q = 1$ in equation (1.3), we get the following form of k-Mittag-Leffler function as follows:

$$ {}_pE^{\eta,1}_{k,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{{}_p(\eta)_{n,k}}{{}_p\Gamma_k(n\lambda+\mu)} \frac{z^n}{n!}. \tag{1.4}$$

2. On setting $p = k$ in equation (1.3), we get the following form of k-Mittag-Leffler function as follows [3]:

$$ {}_kE^{\eta,q}_{k,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{{}_k(\eta)_{nq,k}}{{}_k\Gamma_k(n\lambda+\mu)} \frac{z^n}{n!} = GE^{\eta,q}_{k,\lambda,\mu}(z). \tag{1.5}$$

3. On setting $p = k, q = 1$ in equation (1.3), we get the following form of k-Mittag-Leffler function as follows [4]:

$$ {}_kE^{\eta,1}_{k,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(n\lambda+\mu)} \frac{z^n}{n!} = E^{\eta}_{k,\lambda,\mu}(z). \tag{1.6}$$

4. On setting $p = k, k = 1$ in equation (1.3), we get the following form of Mittag-Leffler function as follows [5]:

$$ {}_1E^{\eta,q}_{1,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{(\eta)_{nq}}{\Gamma(n\lambda+\mu)} \frac{z^n}{n!} = E^{\eta,q}_{\lambda,\mu}(z). \tag{1.7}$$

5. On setting $p = k, q = 1, k = 1$ in equation (1.3), we get the following form of Mittag-Leffler function as follows [6]:

$$ {}_1E^{\eta,1}_{1,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{(\eta)_{n}}{\Gamma(n\lambda+\mu)} \frac{z^n}{n!} = E^{\eta}_{\lambda,\mu}(z). \tag{1.8}$$

6. On setting $p = k, q = 1, k = 1$ and $\eta = 1$ in equation (1.3), we get the following form of Mittag-Leffler function as follows [4]:

$$ {}_1E^{1,1}_{1,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{z^n}{\Gamma(n\lambda+\mu)} = E_{\lambda,\mu}(z). \tag{1.9}$$

7. On setting $p = k, q = 1, k = 1, \eta = 1$ and $\mu = 1$ in equation (1.3), we get the following form of Mittag-Leffler function as follows [7]:

$$ {}_1E^{1,1}_{1,\lambda,\mu}(z) = \sum_{0}^{\infty} \frac{z^n}{\Gamma(n\lambda + 1)} = E_\lambda(z). \tag{1.10} $$

2. Fractional Kinetic equations

The fractional kinetic equations play important role in the field of applied sciences and engineering. There is a rich literature available describing the extensions and generalizations of fractional kinetic equations involving many special functions (see for example [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23])

If an arbitrary reaction is described by a time dependent quantity $N = N(t)$, then the fractional differential equation between rate of change of the reaction, the destruction rate and the production rate of N was established by Haubold and Mathai in [16], is given as follows:

$$ \frac{dN}{dt} = -d(N_t) + p(N_t), \tag{2.1} $$

where $N = N(t)$ represents the rate of reaction, $d = d(N)$ the rate of destruction, $p = p(N)$ the rate of production and N_t denotes the function defined by $N_t(t^*) = N(t-t^*), t^* > 0$.

When spatial fluctuations and inhomogeneities in the quantity $N(t)$ are neglected, then we have the following special case of equation (2.1) as:

$$ \frac{dN}{dt} = -c_i N_i(t), \tag{2.2} $$

where $N_i(t = 0) = N_0$ is the number density of the species i at time $t = 0$ and $c_i > 0$.

On removing the index i and integrating the standard kinetic equation (2.2), we have

$$ N(t) - N_0 = -c_0 D_t^{-1} N(t), \tag{2.3} $$

where c is a constant and ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville fractional integral operator ${}_0D_t^{-\nu}$ defined as

$$ {}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) ds, \qquad (t > 0, R(\nu) > 0). \tag{2.4} $$

The fractional generalization of the standard kinetic equation (2.3) is given by Haubold and Mathai [16] as follows:

$$N(t) - N_0 = -c^\nu {}_0D_t^{-\nu} N(t), \tag{2.5}$$

The solution of (2.5) is given as follows [16]:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}. \tag{2.6}$$

Further, Saxena and Kalla [20] considered the the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t) \qquad (\Re(\nu) > 0, c > 0), \tag{2.7}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and $f \in \mathcal{L}(0, \infty)$.

By applying the Laplace transform to (2.7) (see [24]),

$$L\{N(t); p\} = N_0 \frac{F(p)}{1 + c^\nu p^{-\nu}} = N_0 \left(\sum_{n=0}^{\infty} (-c^\nu)^n p^{-\nu n} \right) F(p), \qquad \left(n \in N_0, \left| \frac{c}{p} \right| < 1 \right). \tag{2.8}$$

Let $f(t)$ be a real or complex valued function of variable t and p is a real or complex parameter, then Laplace transform of $f(t)$ is defined as (see [25])

$$F(p) = L\{f(t); p\} = \int_0^\infty e^{-pt} f(t) dt, \qquad (\Re(p) > 0). \tag{2.9}$$

3. Solution of generalized fractional kinetic equations

In this section, we will study the solution of the generalized fractional kinetic equations by considering generalized $p - k$ Mittag-Leffler function.

Theorem 1. *If $a > 0, d > 0, \nu > 0,\ p, k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ and $q \in (0, 1) \cup \mathbb{N}$, then the equation*

$$N(t) - N_{0\,p}E_{k,\lambda,\mu}^{\eta,q}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{3.1}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k} \Gamma(\nu n + 1)}{{}_p\Gamma_k(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.2}$$

Proof. Taking the Laplace transform to the both sides of equation (3.1) gives

$$L\left\{N(t);p\right\} = N_0 L\left\{{}_pE^{\eta,q}_{k,\lambda,\mu}(d^\nu t^\nu);p\right\} - a^\nu L\left\{{}_0D_t^{-\nu}N(t);p\right\}, \tag{3.3}$$

now using the equation (2.9), we have

$$N(p) = N_0 \int_0^\infty e^{-pt}{}_pE^{\eta,q}_{k,\lambda,\mu}(d^\nu t^\nu)dt - a^\nu p^{-\nu}N(p), \tag{3.4}$$

further using equation (1.3) and interchanging the order of summation and integration, we have

$$N(p) + a^\nu p^{-\nu}N(p) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu)^n}{n!} \int_0^\infty e^{-pt}t^{\nu n}dt, \tag{3.5}$$

which can be written as

$$N(p)(1 + a^\nu p^{-\nu}) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu)^n}{n!} \frac{\Gamma(\nu n+1)}{p^{\nu n+1}}, \tag{3.6}$$

which on simplification gives

$$N(p) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu)^n}{n!} \left\{p^{-(\nu n+1)} \sum_{r=0}^\infty \left[-\left(\frac{p}{a}\right)^{-\nu}\right]^r\right\}. \tag{3.7}$$

Now taking Laplace inverse of equation (3.7) and by using the result given as follows:

$$L^{-1}\left\{p^{-\nu};t\right\} = \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad (\Re(\nu) > 0), \tag{3.8}$$

we have

$$L^{-1}\left\{N(p)\right\} = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu)^n}{n!} L^{-1}\left\{\sum_{r=0}^\infty (-1)^r a^{\nu r} p^{-(\nu n+\nu r+1)}\right\}, \tag{3.9}$$

after simplification, we have

$$N(t) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu)^n}{n!} \left\{\sum_{r=0}^\infty a^{\nu r} \frac{(-1)^r t^{\nu n+\nu r}}{\Gamma(\nu n+\nu r+1)}\right\}, \tag{3.10}$$

which can be written as

$$N(t) = N_0 \sum_0^\infty \frac{{}_p(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu t^\nu)^n}{n!} \left\{\sum_{r=0}^\infty \frac{(-a^\nu t^\nu)^r}{\Gamma(\nu n+\nu r+1)}\right\}, \tag{3.11}$$

the above equation (3.11) gives the required result (3.2).

□

Theorem 2. *If $d>0,\nu>0,\ p,k\in\mathbb{R}^+-0;\ \lambda,\mu,\eta\in\mathbb{C}\setminus k\mathbb{Z}^-;\ \Re(\lambda)>0,\Re(\mu)>0,\Re(\eta)>0$ and $q\in(0,1)\cup\mathbb{N}$, then the equation*

$$N(t)-N_{0p}E^{\eta,q}_{k,\lambda,\mu}(d^\nu t^\nu)=-d^\nu{}_0D_t^{-\nu}N(t) \tag{3.12}$$

has the following solution

$$N(t)=N_0\sum_0^\infty\frac{{}_p(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)}\frac{(d^\nu t^\nu)^n}{n!}E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.13}$$

Proof. On setting $a=d$, the Theorem 1 reduce to the Theorem 2, so details are omitted here. □

Theorem 3. *If $d>0,\nu>0,\ p,k\in\mathbb{R}^+-0;\ \lambda,\mu,\eta\in\mathbb{C}\setminus k\mathbb{Z}^-;\ \Re(\lambda)>0,\Re(\mu)>0,\Re(\eta)>0$ and $q\in(0,1)\cup\mathbb{N}$, then the equation*

$$N(t)-N_{0p}E^{\eta,q}_{k,\lambda,\mu}(t)=-d^\nu{}_0D_t^{-\nu}N(t) \tag{3.14}$$

has the following solution

$$N(t)=N_0\sum_0^\infty\frac{{}_p(\eta)_{nq,k}t^n}{{}_p\Gamma_k(n\lambda+\mu)}E_{\nu,\ n+1}(-d^\nu t^\nu). \tag{3.15}$$

Proof. Proof of Theorem 3 is similar to Theorem 1, so details are omitted here. □

3.1. Special Cases

By setting different values of the parameters, certain interesting results are obtained as follows:

On setting $q=1$, results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 1. *If $a>0,d>0,\nu>0,\ p,k\in\mathbb{R}^+-0;\ \lambda,\mu,\eta\in\mathbb{C}\setminus k\mathbb{Z}^-;\ \Re(\lambda)>0,\Re(\mu)>0,\Re(\eta)>0$, then the equation*

$$N(t)-N_{0p}E^{\eta,1}_{k,\lambda,\mu}(d^\nu t^\nu)=-a^\nu{}_0D_t^{-\nu}N(t) \tag{3.16}$$

has the following solution

$$N(t)=N_0\sum_0^\infty\frac{{}_p(\eta)_{n,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)}\frac{(d^\nu t^\nu)^n}{n!}E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.17}$$

Corollary 2. *If $d>0,\nu>0,\ p,k\in\mathbb{R}^+-0;\ \lambda,\mu,\eta\in\mathbb{C}\setminus k\mathbb{Z}^-;\ \Re(\lambda)>0,\Re(\mu)>0,\Re(\eta)>0$, then the equation*

$$N(t)-N_{0p}E^{\eta,1}_{k,\lambda,\mu}(d^\nu t^\nu)=-d^\nu{}_0D_t^{-\nu}N(t) \tag{3.18}$$

has the following solution

$$N(t) = N_0 \sum_{0}^{\infty} \frac{{}_p(\eta)_{n,k}\Gamma(\nu n+1)}{{}_p\Gamma_k(n\lambda+\mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.19}$$

Corollary 3. *If* $d > 0, \nu > 0,\ p, k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \backslash k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0,$ *then the equation*

$$N(t) - N_{0p}E^{\eta,1}_{k,\lambda,\mu}(t) = -d^\nu{}_0D_t^{-\nu}N(t) \tag{3.20}$$

has the following solution

$$N(t) = N_0 \sum_{0}^{\infty} \frac{{}_p(\eta)_{n,k}t^n}{{}_p\Gamma_k(n\lambda+\mu)} E_{\nu,\ n+1}(-d^\nu t^\nu). \tag{3.21}$$

When $p = k$ then the results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 4. *If* $a > 0, d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then the equation*

$$N(t) - N_{0k}E^{\eta,q}_{k,\lambda,\mu}(d^\nu t^\nu) = -a^\nu{}_0D_t^{-\nu}N(t) \tag{3.22}$$

has the following solution

$$N(t) = N_0 \sum_{0}^{\infty} \frac{{}_k(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_k\Gamma_k(n\lambda+\mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.23}$$

Corollary 5. *If* $d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then the equation*

$$N(t) - N_{0k}E^{\eta,q}_{k,\lambda,\mu}(d^\nu t^\nu) = -d^\nu{}_0D_t^{-\nu}N(t) \tag{3.24}$$

has the following solution

$$N(t) = N_0 \sum_{0}^{\infty} \frac{{}_k(\eta)_{nq,k}\Gamma(\nu n+1)}{{}_k\Gamma_k(n\lambda+\mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.25}$$

Corollary 6. *If* $d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then the equation*

$$N(t) - N_{0k}E^{\eta,q}_{k,\lambda,\mu}(t) = -d^\nu{}_0D_t^{-\nu}N(t) \tag{3.26}$$

has the following solution

$$N(t) = N_0 \sum_{0}^{\infty} \frac{{}_k(\eta)_{nq,k}t^n}{{}_k\Gamma_k(n\lambda+\mu)} E_{\nu,\ n+1}(-d^\nu t^\nu). \tag{3.27}$$

When $p = k, q = 1$ then the results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 7. *If $a > 0, d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$, then the equation*

$$N(t) - N_{0k}E^{\eta,1}_{k,\lambda,\mu}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{3.28}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{{}_k(\eta)_{n,k}\Gamma(\nu n + 1)}{{}_k\Gamma_k(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.29}$$

Corollary 8. *If $d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$, then the equation*

$$N(t) - N_{0k}E^{\eta,1}_{k,\lambda,\mu}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.30}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{{}_k(\eta)_{n,k}\Gamma(\nu n + 1)}{{}_k\Gamma_k(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.31}$$

Corollary 9. *If $d > 0, \nu > 0,\ k \in \mathbb{R}^+ - 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$, then the equation*

$$N(t) - N_{0k}E^{\eta,1}_{k,\lambda,\mu}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.32}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{{}_k(\eta)_{n,k} t^n}{{}_k\Gamma_k(n\lambda + \mu)} E_{\nu,\ n+1}(-d^\nu t^\nu). \tag{3.33}$$

When $p = k, k = 1$ then the results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 10. *If $a > 0, d > 0, \nu > 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ and $q \in (0, 1) \cup \mathbb{N}$, then the equation*

$$N(t) - N_0 E^{\eta,q}_{\lambda,\mu}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{3.34}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{(\eta)_{nq}\Gamma(\nu n + 1)}{\Gamma(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.35}$$

Corollary 11. *If* $d > 0, \nu > 0,\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then the equation*

$$N(t) - N_0 E_{\lambda,\mu}^{\eta,q}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.36}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{(\eta)_{nq}\Gamma(\nu n + 1)}{\Gamma(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.37}$$

Corollary 12. *If* $d > 0, \nu > 0,\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *and* $q \in (0,1) \cup \mathbb{N}$*, then the equation*

$$N(t) - N_0 E_{\lambda,\mu}^{\eta,q}(t) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.38}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{(\eta)_{nq} t^n}{\Gamma(n\lambda + \mu)} E_{\nu,\ n+1}(-d^\nu t^\nu). \tag{3.39}$$

When $p = k, q = 1, k = 1$ then the results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 13. *If* $a > 0, d > 0, \nu > 0;\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$ *then the equation*

$$N(t) - N_0 E_{\lambda,\mu}^{\eta}(d^\nu t^\nu) = -a^\nu {}_0D_t^{-\nu} N(t) \tag{3.40}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{(\eta)_n\Gamma(\nu n + 1)}{\Gamma(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-a^\nu t^\nu). \tag{3.41}$$

Corollary 14. *If* $d > 0, \nu > 0,\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0$*, then the equation*

$$N(t) - N_0 E_{\lambda,\mu}^{\eta}(d^\nu t^\nu) = -d^\nu {}_0D_t^{-\nu} N(t) \tag{3.42}$$

has the following solution

$$N(t) = N_0 \sum_0^\infty \frac{(\eta)_n\Gamma(\nu n + 1)}{\Gamma(n\lambda + \mu)} \frac{(d^\nu t^\nu)^n}{n!} E_{\nu,\ \nu n+1}(-d^\nu t^\nu). \tag{3.43}$$

Corollary 15. *If $d > 0, \nu > 0,\ \lambda, \mu, \eta \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0, \Re(\eta) > 0,$ then the equation*

$$N(t) - N_0 E^{\eta}_{\lambda,\mu}(t) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{3.44}$$

has the following solution

$$N(t) = N_0 \sum_0^{\infty} \frac{(\eta)_n t^n}{\Gamma(n\lambda + \mu)} E_{\nu,\ n+1}(-d^{\nu} t^{\nu}). \tag{3.45}$$

When $p = k, q = 1, k = 1$ and $\eta = 1$ then the results in Theorem 1, Theorem 2 and Theorem 3 reduce to the following form:

Corollary 16. *If $a > 0, d > 0, \nu > 0;\ \lambda, \mu \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0$ then the equation*

$$N(t) - N_0 E_{\lambda,\mu}(d^{\nu} t^{\nu}) = -a^{\nu}{}_0D_t^{-\nu} N(t) \tag{3.46}$$

has the following solution

$$N(t) = N_0 \sum_0^{\infty} \frac{\Gamma(\nu n + 1)(d^{\nu} t^{\nu})^n}{\Gamma(n\lambda + \mu)} E_{\nu,\ \nu n+1}(-a^{\nu} t^{\nu}). \tag{3.47}$$

Corollary 17. *If $d > 0, \nu > 0,\ \lambda, \mu \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0,$ then the equation*

$$N(t) - N_0 E_{\lambda,\mu}(d^{\nu} t^{\nu}) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{3.48}$$

has the following solution

$$N(t) = N_0 \sum_0^{\infty} \frac{\Gamma(\nu n + 1)(d^{\nu} t^{\nu})^n}{\Gamma(n\lambda + \mu)} E_{\nu,\ \nu n+1}(-d^{\nu} t^{\nu}). \tag{3.49}$$

Corollary 18. *If $d > 0, \nu > 0,\ \lambda, \mu \in \mathbb{C} \setminus k\mathbb{Z}^-;\ \Re(\lambda) > 0, \Re(\mu) > 0,$ then the equation*

$$N(t) - N_0 E_{\lambda,\mu}(t) = -d^{\nu}{}_0D_t^{-\nu} N(t) \tag{3.50}$$

has the following solution

$$N(t) = N_0 \sum_0^{\infty} \frac{(\eta)_n t^n}{\Gamma(n\lambda + \mu)} E_{\nu,\ n+1}(-d^{\nu} t^{\nu}). \tag{3.51}$$

4. Conclusion

The fractional kinetic equations are studied involving $p-k$-Mittag-Leffler function. The results obtained are expressed in terms of Mittag-Leffler function. By giving different values to the parameters involved, we get a number of certain interesting results. Due to close relationship of $p-k$-Mittag-Leffler function with the others special functions we can obtain a further generalized fractional kinetic equations which can be very useful in various fields of basic sciences and engineering.

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